Problem Set II: Due TBA

For 216 and 116

Refs: For a more extended reading on these two topics you might consult Acheson's book, and Chapters 5 and 9 of the book "Fluid Mechanics" by Kundu and Cohen.

1) GENERAL DEFORMATION OF A FLUID ELEMENT

Define the rate of strain tensor as $\varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$, where u_i is the i-th component of the velocity (assumed incompressible $\boldsymbol{\partial}\cdot\boldsymbol{u} = 0$) and ∂_j is the j-th component of the spatial gradient. Consider the velocity at two neighboring points $\boldsymbol{x}$ and $\boldsymbol{x}'$, with the separation $\boldsymbol{s} = \boldsymbol{x}' - \boldsymbol{x}$. First-order Taylor expansion of the velocity yields $\boldsymbol{u}' = \boldsymbol{u} + (\boldsymbol{s}\cdot\boldsymbol{\partial})\,\boldsymbol{u}$. Prove and use the equality $\frac{1}{2}(\boldsymbol{s}\cdot\boldsymbol{\partial})\,\boldsymbol{u} = \frac{1}{2}\boldsymbol{\omega}\wedge\boldsymbol{s} + \frac{1}{2}\boldsymbol{\partial}\,(\boldsymbol{s}\cdot\boldsymbol{u})$ to demonstrate:

$$\boldsymbol{u}' = \boldsymbol{u} + \frac{1}{2}\boldsymbol{\omega}\wedge\boldsymbol{s} + \frac{1}{2}\boldsymbol{\partial}_s\,(\varepsilon_{ij}s_i s_j)\ . \tag{1}$$

The second term on the r.h.s. represent a local rigid-body rotation, which illustrates the meaning of the vorticity vector $\boldsymbol{\omega}$ as a measure of the local spinning of fluid elements.

As for the last term on the r.h.s., we want to show that it represents a pure straining motion. To that purpose, use incompressibility to show that the rate of strain tensor is traceless. Using this property and diagonalizing the quadratic form $\varepsilon_{ij}s_i s_j$, convince yourself that iso-surfaces of the quadratic form are hyperboloids and the associated gradients $\boldsymbol{\partial}_s\,(\varepsilon_{ij}s_i s_j)$ correspond to pure strain, i.e. stretching/squashing in perpendicular directions without any overall rotation.

2) RANKINE VORTEX AS A SIMPLE MODEL FOR TORNADOES

Consider an axisymmetric flow with tangential velocity $\boldsymbol{u} = u(r)\hat{\boldsymbol{e}}_\phi$, where $(r,\,\phi,\,z)$ are cylindrical coordinates. The velocity depends on the radial distance as $u(r) = \Omega a^2/r$ for $r \geq a$ and $u(r) = \Omega r$ for $r \leq a$ where a is a radius of the vortex. Calculate the corresponding vorticity field. Note that the flow is irrotational outside the vortex, i.e. $r > a$.

Real vortices are typically characterized by small vortex cores where the vorticity is concentrated, whilst outside the core the flow is essentially irrotational. The core is not usually circular, nor is the vorticity uniform. In these two respects the Rankine vortex is only a simplified model of real vortices.

Use the Euler equations $(\boldsymbol{u}\cdot\boldsymbol{\partial})\,\boldsymbol{u} = -\boldsymbol{\partial}p/\rho - g\hat{z}$, where g is the gravitational acceleration and ρ is the (constant) density to derive that

$$p(r) = \begin{cases} p_0 - \frac{\rho\Omega^2 a^4}{2r^2} - \rho g z, & r \geq a \\ p_0 - \rho\Omega^2 a^2 + \frac{\rho\Omega^2 r^2}{2} - \rho g z, & r \leq a \end{cases} \tag{2}$$

where p_0 is the atmospheric pressure, i.e. the value at large r and at the surface of the fluid $z = 0$. Conclude that the pressure at $z = 0$ in the center of the vortex is lower than the atmospheric pressure by an amount $\rho\Omega^2 a^2$. The depression in the core of a tornado is a major cause of its destructive effects.

Deduce that the free surface of the liquid at $r = 0$ is at a depth $\Omega^2 a^2/g$ below the surface at infinity (hence the dimples when a cup of liquid is stirred by a rotating spoon).

For 216 and 116 (cont.)

3) a) Use the condition $\underline{\nabla} \cdot v = 0$ to solve for the pressure $P(\underline{x})$. Insert this into the Euler equation to get an evolution equation for $\underline{v}$ $(\underline{x}, t)$ only. Discuss the structure of the equation. How does it differ from that of the usual way the Euler equation is written?

b) Now, let the flow be compressible, with equation of state $P = P(\rho)$. Use the continuity equation and the Euler equation to formally obtain $P(\underline{x}, t)$. Obtain the compressible Euler equation and discuss its structure. Why is it non-local in time?

c) What do you generally conclude about the physics and role of pressure from a), b), above?

For 216

4) Work thru the full calculation of the drag in Stokesian hydrodynamics. Give all details.